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Exact solution of Kauffman's model with connectivity one

H Flyvbjerg and N J Kjær

The Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

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Abstract Kauffman's model is a randomly assembled network of Boolean automata. Each automaton receives inputs from at most K other automata. Its state at discrete time $t+1$ is determined by a randomly chosen, but fixed, Boolean function of the K inputs at time t. The resulting quenched, random dynamics of the network demonstrates two phases: a frozen and a chaotic phase. We give an exact solution of the model for connectivity $K = 1$, valid everywhere in the frozen phase and at a critical point, valid for finite as well as for infinite networks. We discuss the network's critical behaviour and finite-size effects. The results for the frozen phase presented here complement recent exact results for the chaotic phase obtained for $K = \infty$.

1. Introduction

Exactly sovable models enjoy a special status wherever mathematical modelling is done. Though rare, they play a central role in our understanding both of the systems modelled and of all related, less solvable models. They often also serve as non-trivial starting points in approximation schemes, for non-solvable models. As such, the existence of a solvable model may be of crucial value for further model building.

Simplified models are a vastly more abundant commodity. As such, Kauffman's model was originally introduced to model the complex genetic regulatory system that guides cell differentiation in embryonic development [1-4]. It has recently received increased attention from workers in the theory of automata, in neural networks and in disordered systems [5-18]. Viewed as a network of automata, the model is non-homogeneous and infinite dimensional, which makes it more complex than the cellular automata studied by Wolfram and others [19]. Viewed as a formal neural network the arbitrary dependence on inputs makes it more complicated than, for example, the Little-Hopfield model [20-22]. Viewed as a disordered system, the absence of temperature and the deterministic dynamics makes the model more tractable than spin glasses, e.g. systems of finite size have a multivalley structure [14].

A number of properties of Kauffman's model with connectivity $K = \infty$ have recently been obtained exactly [15]. The highly non-trivial results describe the model deep in its chaotic phase (see below). In the present paper we solve the model for connectivity $K = 1$. The equally non-trivial results describe the model in its frozen phase. By varying a parameter q of the model we can follow the behaviour of our exact results all the way to the phase boundary. We have reasons to believe that our exact solution can be made the starting point for an approximation scheme describing the model for any connectivity K anywhere in the frozen phase, and possibly across the phase boundary and slightly into the chaotic phase, i.e. the critical behavour might be described in this scheme.

Now let us describe Kauffman's model. Kauffman's model consists of a network of N Boolean variables σ_i, $i=1,\ldots,N$, which develop in discrete time according to a deterministic dynamic law: the value of σ_i at time $t+1$ depends on the value of at most K variables at time t:

$$\sigma_i(t+1)=f_i(\sigma_{j_1(i)}(t), \sigma_{j_2(i)}(t), \ldots, \sigma_{j_K(i)}(t)). \tag{1.1}$$

Like N, the connectivity K is a parameter of the model. A sample of such a network is defined by giving the connection graph $(j_1(i),\ldots,j_K(i))_{i=1,\ldots,N}$ and the Boolean functions $(f_i)_{i=1,\ldots,N}$. The connection graph is obtained by choosing $j_1(i)$, $j_2(i),\ldots,j_K(i)$ at random between 1 and N, independently of each other and of choices for other values of i. So the connection graph is a random, directed graph with at least one and at most K different edges $j_1(i),\ldots,j_K(i)$ terminating at the vertex i; $i=1,\ldots,N$. The Boolean functions f_i are also chosen at random: a third parameter p is introduced by choosing

$$f_i(\sigma_1,\ldots,\sigma_K)=\begin{cases}1 & \text{with probability } p\\ 0 & \text{with probability } 1-p\end{cases} \tag{1.2}$$

for any input $(\sigma_1,\ldots,\sigma_K)$, with no regard for the value of f_i for other inputs. Once a sample is defined, its connection graph and Boolean functions are kept fixed, and the consequences of the resulting quenched, random dynamics may be studied. Usually only sample-averaged quantitites are of interest.

In § 2 we describe our strategy for solving the model. It is shown that typically only a few, rather short, so-called information conserving loops in the connection graph determine the partitioning of configuration space into basins of attraction for limit cycles. In § 3 we calculate the probability P for a given distribution of loops in the connection graph. Section 4 gives the probability Q for information conserving loops. Q is a fundamental quantity in our calculations on the model, and we believe that most quantities one may think of can be calculated along the lines we use, starting with our expression for Q. In § 5 we obtain a rapidly converging series for the probability $g(W)$ that a randomly chosen configuration of the variables $(\sigma_i)_{i=1,\ldots,N}$ belongs to a basin of attraction of relative size W. With Y_P denoting the probability that P randomly chosen configurations belong to the same basin of attraction, we also obtain the probability distribution Π_P for Y_P as a rapidly converging series. Terms in these series are calculated one by one, starting with the dominant and simpler ones. This rather pedantic section spells out what goes on in a typical sample of the network. We suggest the reader reads as far as he benefits from it, and then jumps to the end of the section. Our technique is vaguely reminiscent of the high-temperature expansion of lattice spin theories, or the strong coupling expansion of lattice gauge theories. Our series converge, however, in the entire phase under study. In § 6 we even find that in any interval not containing zero on the W or Y_P axis, the series for $g(W)$ and $\Pi_P(Y_P)$ contain only a finite number of terms. This makes it possible to derive very narrow bounds on expectation values computed from the exact probability distributions $g(W)$ and $\Pi_P(Y_P)$, which is done in § 7. Section 8 discusses a critical point found at the end of a parameter interval, much like the critical point of one-dimensional spin systems at zero temperature. Section 9 discusses effects of finite system size N. Section 10 contains our conclusions.

While the work presented here was progressing, we received a preprint by Hilhorst and Nijmeijer [23], which amongst other interesting results for Kauffman's model gives an exact expression for the time evolution of the overlap between two configurations in the case of connectivity $K = 1$.

2. Strategy for a solution

Consider Kauffman's model with connectivity $K = 1$. Let N be the number of Boolean variables in the model. Since each variable receives its input from one variable chosen at random amongst all N variables, the connection graph is determined by a random map of a set of N points into itself [1]. Several variables can receive input from the same variable; let us call it their immediate ancestor, and call them its immediate descendants. By 'descendant' we shall mean any immediate descendant of an immediate descendant . . . of an immediate descendant. Similarly, by 'ancestor' we shall mean any immediate ancestor of an immediate ancestor . . . of an immediate ancestor. Since every variable has one and only one immediate ancestor, but can have any number of immediate descendants from 0 to N, any of the N variables of Kauffman's model with connectivity $K = 1$ lies either on a generation tree or on a loop. A loop occurs when a variable is amongst its own ancestors. Loops can occur with any length between 1 and N, the length being the number of variables on the loop. Since all variables have ancestors, and there is only a finite number N of variables, at least one loop occurs. All trees are rooted in loops. So much for the connection graph.

The value of any variable σ_i at time $t+1$ is determined by the value of its immediate ancestor $\sigma_{j(i)}$ at time t via a Boolean function f_i:

$$\sigma_i(t+1) = f_i(\sigma_{j(i)}(t)). \tag{2.1}$$

So, no matter what the initial configuration $(\sigma_i(0))_{i=1,\ldots,N}$ is, after at most $N-1$ time steps the information contained in $(\sigma_i(0))_{i\in\text{trees}}$ is lost, and $(\sigma_i(t))_{i=1,\ldots,N}$ is, for $t \geq N$, determined by the functions $(f_i)_{i=1,\ldots,N}$ and $(\sigma_i(0))_{i\in\text{loops}}$. Furthermore, any information in $(\sigma_i(0))_{i\in\text{loops}}$ which is placed on loops containing one or more of the constant functions $f_i(\sigma) = 1$ for $\sigma = 0, 1$ or $f_i(\sigma) = 0$ for $\sigma = 0, 1$ is also lost after at most N time steps, and $(\sigma_i(t))_{i\in\text{loop}}$ is, if the loop contains a constant function f_i, for $t \geq N$ entirely determined by the functions $(f_i)_{i\in\text{loop}}$. Thus, in our search for the probabilistic laws governing the behaviour of the network, we are led to consider only loops in the connection graph, and only such loops that contain no constant functions f_i. For time $t \geq N$ all variables $\sigma_i(t)$ on such loops and on the trees rooted in them, are determined by $(\sigma_i(0))_{i\in\text{loop}}$ and the functions f_i on these loops and trees.

The probability for either non-constant function $f_i(\sigma) = \sigma$ and $f_i(\sigma) = \text{not}(\sigma)$ is $p(1-p)$. So the probability that a loop of length L contains only these functions is $[2p(1-p)]^L$. For any value of p between zero and one this probability is less than 2^{-L}, showing that only short loops have a non-vanishing probability for not containing a constant function. So we have the qualitative result that, even for $N \to \infty$, the configuration $(\sigma_i(t))_{i=1,\ldots,N}$ for $t \geq N$ depends only on a few of the variables $\sigma_i(0)$ in the initial configuration, those on some short loops in the connection graph. Let us call these variables the relevant ones. For a given sample $(f_i, j(i))_{i=1,\ldots,N}$ of the network the number of different cycles in time that the relevant variables can form is also the number of different basins of attraction for this sample. The number of initial configurations that fall on an attractor is just the number of initial configurations with relevant variables falling on the same cycle.

From the scenario just developed, it is obvious that there must be sample-to-sample fluctuations of a number of quantities in the thermodynamic limit $N\to\infty$, simply because the behaviour of a macroscopic number of variables is fully determined by the microscopic, hence fluctuating, number of relevant variables and functions $(\sigma_i, f_i, j(i))_{i\ \text{relevant}}$.

Now let us calculate the probabilities for the situations described above.

3. $P(n)$, the probability for loop distribution $(n_L)_{L=1,2,\ldots}$

Consider the random map of a set of N points into itself. For a sample of this map let n_L denote the number of loops of length L, $L=1,2,3,\ldots$. Below we prove that the probability $P(n)$ for the occurrence of a given distribution $(n_L)_{L=1,2,\ldots}$ of loop lengths is

$$P(n)=\frac{\hat{n}}{N}\frac{N!}{(N-\hat{n})!\,N^{\hat{n}}}\prod_{L=1}^{\infty}\frac{L^{-n_L}}{n_L!} \tag{3.1}$$

where

$$\hat{n}=\sum_{L=1}^{\infty} n_L L. \tag{3.2}$$

For later convenience we have used infinity instead of N as the upper limits in the product in (3.1) and the sum in (3.2). This does not alter their values since $1/(N-n)!=0$ for $n=N+1, N+2,\ldots$. Notice also that $P(n)=0$ if $n_L=0$ for all L, so we can include this value for n in sums over n without altering their values.

The proof of (3.1) is as follows. The number of different ways we can partition a set of N distinguishable objects into n_L sets containing L objects, $L=1,2,3,\ldots$, and the complement of these n_L sets, is

$$\frac{N!}{(N-\hat{n})!\prod_{L=1}^{\infty} n_L!(L!)^{n_L}}. \tag{3.3}$$

When the random map is applied to a set of N objects partitioned this way, the probability that each of the n_L sets of L objects map into themselves to form a loop is

$$\prod_{L=1}^{\infty}\left(\frac{(L-1)!}{N^L}\right)^{n_L}=\frac{1}{N^{\hat{n}}}\prod_{L=1}^{\infty}[(L-1)!]^{n_L}. \tag{3.4}$$

Finally, the probability that the random map applied to a set of N objects forms loops that all are entirely inside a subset $\mathscr{L}$ of $\hat{n}$ objects is just the probability that repeated application of the random map to any element in the complement $C\mathscr{L}$ iterates to an element in $\mathscr{L}$ (where it stays under further iterations, when we have assumed that $\mathscr{L}$ is entirely made up of loops). This probability is $\hat{n}/N$. This is not obvious, but calling this probability $P(\hat{n};N)$ and applying the random map repeatedly to any element in $C\mathscr{L}$, one gets the recursion relation

$$P(\hat{n};N)=(N-\hat{n}-1)!\,\hat{n}\sum_{L=1}^{N-\hat{n}}\frac{P(\hat{n}+L;N)}{N^L(N-\hat{n}-L)!}. \tag{3.5}$$

Equation (3.5) expresses that any element x in $C\mathscr{L}$ by L applications of the random map, $L\in\{1,2,\ldots,N-\hat{n}\}$, should iterate to an element in $\mathscr{L}$ and that any element y

in $C\mathscr{L}\backslash\{\text{trajectory of } x\}$ should iterate to an element in $\mathscr{L}\cup\{\text{trajectory of } x\}$. Equation (3.5) is solved by

$$P(\hat{n};\, N)=\hat{n}/N. \tag{3.6}$$

The product of the combinatorial factor (3.3) and the probabilities (3.4) and (3.6) gives (3.1).

Since at least one loop must occur in the random map, $P(n)$ is a normalised probability:

$$\sum_{n_1,n_2,\ldots=0}^{\infty} P(n)=1. \tag{3.7}$$

This may also be shown directly: (3.1) and (3.2) give

$$\begin{aligned} P(n)&=\int_0^{\infty} \mathrm{d}x \frac{x}{N}\frac{N!}{(N-x)!\,N^x}\,\delta\Big(x-\sum_L n_L L\Big)\prod_{L=1}^{\infty}\frac{L^{-n_L}}{n_L!}\\ &=\int_0^{\infty} \mathrm{d}x \frac{x}{N}\frac{N!}{(N-x)!\,N^x}\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{2\pi}\exp(-\mathrm{i}\alpha x)\prod_{L=1}^{\infty}\frac{[(1/L)\exp(\mathrm{i}\alpha L)]^{n_L}}{n_L!}. \end{aligned} \tag{3.8}$$

Using

$$\sum_{n_1,n_2,\ldots=0}^{\infty}\prod_{L=1}^{\infty}\frac{[(1/L)\exp(\mathrm{i}\alpha L)]^{n_L}}{n_L!}=\frac{1}{1-\exp(\mathrm{i}\alpha)} \tag{3.9}$$

we find

$$\sum_{n_1,n_2,\ldots=0}^{\infty} P(n)=\sum_{\hat{n}=0}^{N}\frac{\hat{n}}{N}\frac{N!}{(N-\hat{n})!\,N^{\hat{n}}}=1. \tag{3.10}$$

For $N\to\infty$ (3.1) becomes, to leading order in N,

$$P(n)=\frac{\hat{n}}{N}\exp(-\hat{n}^2/2N)\prod_{L=1}^{\infty}\frac{(1/L)^{n_L}}{n_L!}. \tag{3.11}$$

The exponent in (3.11) makes $\hat{n}\gg\sqrt{N}$ highly improbable. The factor $\hat{n}$ is linear both in n_L and in L. The final product, however, strongly suppresses large values of n_L by a factor $1/n_L!$ and large values of L by a factor L^{-n_L}. The balance of this is that $P(n)=\mathrm{O}(1/N)$ in two cases. One is for $\hat{n}=\mathrm{O}(1)$. The other is for $n_L=0$ for all L larger than O(1) except for a single value of L, which must be less than $\mathrm{O}(\sqrt{N})$, and have $n_L=1$. For all other distributions $(n_L)_{L=1,2,\ldots}$ of loop lengths is $P(n)<\mathrm{O}(1/N)$.

The expectation value for n_L can be calculated in a way similar to that which leads to (3.10). One finds

$$\begin{aligned} \bar{n}_{L_0}&=\sum_{n_1,n_2,\ldots=0}^{\infty} n_{L_0}P(n)=\frac{1}{L_0}\frac{N!}{(N-L_0)!\,N^{L_0}}\\ &=\frac{1}{L_0}\exp(-L_0^2/2N)+\mathrm{O}(1/N). \end{aligned} \tag{3.12}$$

We see that $\bar{n}_1\simeq 1$, $\bar{n}_L\lesssim 1/L$ and $\bar{n}_L\approx 0$ for $L>\mathrm{O}(\sqrt{N})$.

4. $Q(m)$, the probability for distribution $(m_L)_{L=1,2,\ldots}$ of information conserving loops

Let $Q(m)$ denote the probability of having $(m_L)_{L=1,2,\ldots}$ information conserving loops of length L, $L=1,2,\ldots$, in the connection graph of Kauffman's model with $K=1$. Below we prove that

$$Q(m)=\frac{N!}{(N-\hat{m})!\,N^{\hat{m}}}[2p(1-p)]^{\hat{m}}[1-2p(1-p)(1-\hat{m}/N)]\prod_{L=1}^{\infty}\frac{L^{-m_L}}{m_L!} \tag{4.1}$$

where

$$\hat{m}=\sum_{L=1}^{\infty} m_L L. \tag{4.2}$$

For $N\to\infty$ to leading order in N, (4.1) becomes

$$\begin{aligned} Q(m)&=[p^2+(1-p)^2][2p(1-p)]^{\hat{m}}\prod_{L=1}^{\infty}\frac{1}{m_L!\,L^{m_L}} \\ &=[p^2+(1-p)^2]\prod_{L=1}^{\infty}\left(\frac{[2p(1-p)]^L}{L}\right)^{m_L}(m_L!)^{-1}. \end{aligned} \tag{4.3}$$

Notice that $Q(m)$ in (4.3) is a product of independent probability distributions, one for each loop length L.

The result (4.1) may be found in the following way: in order to have m_L information conserving loops of length L we must have at least m_L loops of length L, i.e. we must have n_L such loops, $n_L\geqslant m_L$, $L=1,2,3,\ldots$. Thus $Q(m)$ is a sum over distributions $(n_L)_{L=1,2,\ldots}$ satisfying $n_L\geqslant m_L$ of the probability $P(n)$ times the probability that exactly m_L of the n_L loops of length L are information conserving. The probability that a loop of length L is information conserving is the probability that all of its functions f_i are 'identity' or 'negation', i.e. $[2p(1-p)]^L$. So the probability that a loop does not conserve information is $1-[2p(1-p)]^L$. Thus

$$\begin{aligned} Q(m)=\sum_{n_1\geqslant m_1}\sum_{n_2\geqslant m_2}\cdots P(n)\prod_{L=1}^{\infty}\binom{n_L}{m_L}[2p(1-p)]^{Lm_L} \\ \times\{1-[2p(1-p)]^L\}^{n_L-m_L}. \end{aligned} \tag{4.4}$$

Inserting (3.1) in (4.4) and proceeding in a way similar to the one leading to (3.10), we arrive at the formula (4.1).

For $N\to\infty$ the factor $N!/[(N-\hat{m})!\,N^{\hat{m}}]$ in (4.1) equals $\exp(-\hat{m}^2/2N)$ to leading order in N^{-1}. However, since $2p(1-p)\leqslant\frac{1}{2}$ for $p\in[0,1]$, the factor $[2p(1-p)]^{\hat{m}}$ in (4.1) makes $Q(m)$ vanish unless $\hat{m}=\mathrm{O}(1)$. So to leading order in N^{-1} we have (4.3). A useful formula is

$$\sum_{\substack{m \\ \hat{m}\ \text{fixed}}} Q(m)=[p^2+(1-p)^2][2p(1-p)]^{\hat{m}}. \tag{4.5}$$

Q is a normalised probability distribution:

$$\sum_m Q(m)=\prod_{L=1}^{\infty}\sum_{m_L=0}^{\infty}Q(m)=1. \tag{4.6}$$

5. Approximating sequences for $g(W)$ and $\Pi_P(Y_P)$

Introduce in (4.3)

$$q = 2p(1-p) \tag{5.1}$$

where q is the probability that a function f_i is information conserving (i.e. either the identity or negation) and (4.1)-(4.3) are valid for any value of q in $[0, 1]$. Equation (4.3) then becomes

$$Q(m) = (1-q)q^{\hat{m}}\left(\prod_{L=1}^{\infty} m_L!\,L^{m_L}\right)^{-1} \tag{5.2}$$

with $\hat{m}$ as defined in (4.2). For p in $[0, 1]$ q belongs to $[0, \frac{1}{2}]$ and consequently $Q(m)$ rapidly becomes small with increasing $\hat{m}$. So, if we are satisfied with approximate results, we only have to consider a finite set of configurations m of information conserving loops. The approximation involved can be made arbitrarily good by considering larger finite sets of loop configurations. As an example, in this section we shall consider loop configurations m with probability $Q(m) > 0.005$. These configurations of information conserving loops are for any value of q in $[0, \frac{1}{2}]$ contained in the set of ten configurations satisfying $\hat{m} \leqslant 4$.

Within our approximation, we can calculate the relative frequency $f(W)$ with which a basin of weight W occurs or, equivalently, the normalised probability $g(W) = Wf(W)$ that a randomly chosen configuration of variables $(\sigma_i)_{i=1,\ldots,N}$ belongs to a basin of attraction with weight W. Furthermore, since for a given configuration of loops and input functions we calculate the weights W_s themselves (s being an index enumerating the basins of attraction occurring for that configuration), we can also evaluate the quantity

$$Y_P = \sum_s W_s^P \tag{5.3}$$

and the probability distribution Π_P for Y_P. For P a positive integer, Y_P is the probability that P randomly chosen configurations belong to the same basin of attraction.

Now let us show how this is done.

5.1. *The case $m = (0, 0, \ldots)$*

This case occurs with probability

$$Q(0, 0, \ldots) = 1 - q. \tag{5.4}$$

Since it describes a network with no information conserving loops, any initial configuration $(\sigma_i(0))_{i=1,\ldots,N}$ will in time develop to the same configuration. So there is only one basin of attraction, which then has weight $W = 1$. Consequently $Y_P = 1$.

This case contributes to $g(W)$ with an additive term $(1-q)\,\delta\,(W-1)$, and to $\Pi_P(Y_P)$ with an additive term $(1-q)\delta(Y_P - 1)$.

5.2. *The case $m = (1, 0, \ldots)$*

This occurs with probability

$$Q(1, 0, \ldots) = (1-q)q. \tag{5.5}$$

It describes a network with one information conserving loop of length one. The one variable σ_{i_0} in this loop receives input from itself:

$$\sigma_{i_0}(t+1)=f_{i_0}(\sigma_{i_0}(t)). \tag{5.6}$$

$f_{i_0}=$'identity' with probability $\frac{1}{2}$, and consequently any initial configuration $(\sigma_i(0))_{i=1,\ldots,N}$ will develop in time to a configuration determined by the constant value of σ_{i_0}. This value is zero for half of all possible initial configurations, and one for the other half. Consequently configuration space is split with probability $\frac{1}{2}(1-q)q$ into two basins of attraction of equal weights: $W_1=W_2=\frac{1}{2}$. It follows that $Y_P=2(\frac{1}{2})^P$. The attractors in both basins are fixed points.

$f_{i_0}=$ 'not' with probability $\frac{1}{2}$, and consequently any initial configuration will develop in time to a 2-cycle of configurations characterised by $\sigma_{i_0}(t+1)=\text{not}(\sigma_{i_0}(t))$. So there is only one basin of attraction, $W=1$, $Y_P=1$.

This case contributes to $g(W)$ with $\frac{1}{2}(1-q)q(\delta(W-\frac{1}{2})+\delta(W-1))$ and to Π_P with $\frac{1}{2}(1-q)q(\delta(Y_P-2(\frac{1}{2})^P)+\delta(Y_P-1))$.

5.3. *The case $m=(0, 0, 1, 0, \ldots)$*

This occurs with probability

$$Q(0,0,1,0,\ldots)=\tfrac{1}{3}(1-q)q^3 \tag{5.7}$$

and describes a network with one information conserving loop containing three variables; call them σ_{i_1}, σ_{i_2} and σ_{i_3}. They receive inputs from each other such that

$$\sigma_{i_{j+1}}(t+1)=f_{i_{j+1}}(\sigma_{i_j}(t)). \tag{5.8}$$

The functions f_{i_1}, f_{i_2} and f_{i_3} can all be 'identity' (with probability $\frac{1}{8}$), one can be 'not' (with probability $\frac{3}{8}$), two can be 'not' (with probability $\frac{3}{8}$), or all can be 'not' (with probability $\frac{1}{8}$).

When $f_{i_1}=f_{i_2}=f_{i_3}=$ 'identity', any initial configuration $(\sigma_i(0))_{i=1,\ldots,N}$ with $(\sigma_{i_1}(0), \sigma_{i_2}(0), \sigma_{i_3}(0))=(1,1,1)$ develops in time to the same final configuration. Any initial configuration with $(\sigma_{i_1}(0), \sigma_{i_2}(0), \sigma_{i_3}(0))=(1,1,0)$, $(0,1,1)$ or $(1,0,1)$ develops in time to the same 3-cycle in configuration space.

Configurations with $(\sigma_{i_1}(0), \sigma_{i_2}(0), \sigma_{i_3}(0))=(0,0,1)$, $(1,0,0)$ or $(0,1,0)$ develop to another 3-cycle, and configurations with $(\sigma_{i_1}(0), \sigma_{i_2}(0), \sigma_{i_3}(0))=(0,0,0)$ develop to the same final configuration. Thus we have four basins of attraction with weights $W=\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$ and $\frac{1}{8}$. Hence $Y_P=2(\frac{1}{8})^P+2(\frac{3}{8})^P$.

When two of the functions are 'identity' and one is 'not', one finds in a similar way that there are two basins of attraction with weights $W=\frac{3}{4}$ and $\frac{1}{4}$, corresponding to an attractive 6-cycle and 2-cycle, respectively. Hence $Y_P=(\frac{1}{4})^P+(\frac{3}{4})^P$.

When one function f_{i_j} is 'identity' and two are 'not', one finds weights $W=\frac{1}{8}$, $\frac{1}{8}$, $\frac{3}{8}$ and $\frac{3}{8}$, hence $Y_P=2(\frac{1}{8})^P+2(\frac{3}{8})^P$.

When all functions f_{i_j} are 'not', one finds $W=\frac{1}{4}$ and $\frac{3}{4}$, $Y_P=(\frac{1}{4})^P+(\frac{3}{4})^P$.

Summing up, one gets a contribution to $g(W)$ equalling

$$\tfrac{1}{3}(1-q)q^3[\tfrac{1}{8}\delta(W-\tfrac{1}{8})+\tfrac{1}{8}\delta(W-\tfrac{1}{4})+\tfrac{3}{8}\delta(W-\tfrac{3}{8})+\tfrac{3}{8}\delta(W-\tfrac{3}{4})]$$

and a contribution to Π_P equalling

$$\tfrac{1}{6}(1-q)q^3\{\delta[Y_P-2(\tfrac{1}{8})^P-2(\tfrac{3}{8})^P]+\delta[Y_P-(\tfrac{1}{4})^P-(\tfrac{3}{4})^P]\}.$$

Generalising from the case just studied, it is not difficult to see that the way in which a given loop with given input functions breaks configuration space into basins with a certain distribution of weights does not depend in detail on the distribution of input functions, but only on whether the number of functions 'not' occurring is even or odd. If it is even, the L variables on an information conserving loop of length L will go through a cycle with a period that is a divisor in L. If it is odd, they will go through a cycle with a period that is a divisor in $2L$, and not a divisor in L. These observations simplify calculations considerably.

5.4. *The case $m = (1, 0, 1, 0, \ldots)$*

This case describes a network with two information conserving loops, one containing one variable, another containing three. We have just seen how one loop containing three variables can split configuration space into basins of attraction in various ways. When another loop containing one variable σ_{i_0} and function f_{i_0} = 'identity' is added, each basin of attraction created by the loop of length three is split into two basins of equal size, one for each of the two constant values σ_{i_0} can have. If, on the other hand, f_{i_0} = 'not' then $\sigma_{i_0}(t)$ describes a 2-cycle, which can have only one (two different) phases relatively to a cycle with odd (even) period described by the three variables on the loop of length three. So for f_{i_0} = 'not' a basin of attraction created by the loop of length three is not split any further (is split into two basins of equal size) if the attractor is a cycle of odd (even) period. The upshot of this is that the case $m = (1, 0, 1, 0, \ldots)$ contributes to $g(W)$ with

$$\tfrac{1}{3}(1-q)q^4[\tfrac{1}{16}\delta(W-\tfrac{1}{16})+\tfrac{3}{16}\delta(W-\tfrac{1}{8})+\tfrac{3}{16}\delta(W-\tfrac{3}{16})+\tfrac{9}{16}\delta(W-\tfrac{3}{8})]$$

and to Π_P with

$$\tfrac{1}{3}(1-q)q^4\{\tfrac{1}{4}\delta[Y_P-4(\tfrac{3}{16})^P-4(\tfrac{1}{16})^P]+\tfrac{3}{4}\delta[Y_P-2(\tfrac{3}{8})^P-2(\tfrac{1}{8})^P]\}.$$

Going through all ten configurations m satisfying $\hat{m} \leqslant 4$, one finds approximate expressions for $g(W)$ and $\Pi_P(Y_P)$. They are of the form

$$g(W) = \sum_i g_i \delta(W - W_i) \tag{5.9}$$

and

$$\Pi_P(Y_P) = \sum_k \pi_k \delta(Y_P - Y_{P,k}). \tag{5.10}$$

W_i and g_i are given in table 1, $Y_{P,k}$ and π_k in table 2. In actual practice we have used a computer program and treated $\hat{m} \leqslant 23$.

Π_P given in (5.10) is reminiscent of the functions Π_P found for randomly broken objects in the mean-field theory for spin glasses and in Kauffman's model with $K = \infty$ [16]. The mean-field functions have an infinite set of singular points at $Y_P = (1/n)^{P-1}$, $n = 1, 2, \ldots$. The locations of these singular points do not depend on the parameter y occurring in the mean-field theory for spin glasses, just like $Y_{P,k}$ in (5.10) does not depend on the parameter q. The exponent of the singularity, on the other hand, does depend on y, but not on P, just like π_k in (5.10) depends on q, but not on P.

The probability distributions in (5.9) and (5.10) are normalised to unity in principle. When approximated as in this section one finds

$$\int_0^1 \mathrm{d}W g(W) = \sum_i g_i = 1 - q^5 \tag{5.11}$$

Table 1. The weights W_i, where $g(W)$ has support, and the probabilities g_i that a randomly chosen configuration belongs to a basin of attraction with weight W_i. $g_i/(1-q)$ is a polynomial in $x=\frac{1}{2}q$. The table is infinite, so to truncate it somewhere we have dropped all powers of x larger than seven. The polynomials given are exact for $W_i = 1, \frac{3}{4}, \frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \frac{1}{4}$ and $\frac{1}{8}$. The results presented in the figures of this paper included powers of x up to 23. The sum of coefficients of individual powers of x (i.e. x^n) add up to 2^n because $g(W)$ in (5.9) is a normalised probability distribution.

W_i	$g_i/(1-q)$ with $x=\frac{1}{2}q$
1	$1+x+x^2$
$\frac{3}{4}$	x^3
$\frac{1}{2}$	$x+2x^2+2x^3+2x^4$
$\frac{3}{8}$	$x^3+3x^4+2x^5$
$\frac{5}{16}$	$3x^5$
$\frac{1}{4}$	$x^2+3x^3+5x^4+4x^5$
$\frac{3}{16}$	$x^4+5x^5+9x^6+4x^7$
$\frac{5}{32}$	$3x^5+9x^6+6x^7$
$\frac{1}{8}$	$x^3+4x^4+8x^5+8x^6$
$\frac{7}{64}$	$9x^7$
$\frac{3}{32}$	$x^5+13x^6+20x^7$
$\frac{5}{64}$	$3x^6+15x^7$
$\frac{1}{16}$	$x^4+5x^5+13x^6+12x^7$
$\frac{7}{128}$	$9x^7+\cdots$
$\frac{3}{64}$	$2x^6+22x^7+\cdots$
$\frac{5}{128}$	$3x^7+\cdots$
$\frac{1}{32}$	$x^5+6x^6+18x^7+\cdots$
$\frac{3}{128}$	$2x^7+\cdots$
$\frac{1}{64}$	$x^6+7x^7+\cdots$
$\frac{1}{128}$	$x^7+\cdots$
$\vdots$	For $W<\frac{1}{128}$, x occurs only raised to powers larger than seven.

and

$$\int_0^1 \mathrm{d}Y_P\,\Pi_P(Y_P)=\sum_k \pi_k = 1-q^5. \tag{5.12}$$

Since $q\leq\frac{1}{2}$ the approximation of this section misses less than 3% of the total probabilities in g and Π_P.

In the next section we prove that this missed probability is located close to zero, and that our approximate expressions for $g(W)$ and $\Pi_P(Y_P)$ are exact for W and Y_P away from zero. This leads to narrow bounds on $\overline{Y_P}$ and $\overline{Y^n}$ in § 7.

6. Some exact results for $g(W)$ and $\Pi_P(Y_P)$

The heuristic derivation of § 5 illustrates well the mechanisms at play, but misses some useful results for $g(W)$ and $\Pi_P(Y_P)$. They are derived here.

A given distribution $m=(m_L)_{L=1,2,\ldots}$ for the number of information conserving loops of length L leaves us with $\hat{m}$ relevant variables, $\hat{m}=\sum_{L=1}^{\infty} Lm_L$. Each of the $2^{\hat{m}}$ different configurations that these relevant variables can be in, can be realised by $2^{N-\hat{m}}$ different configurations of the entire set of N variables. These $2^{N-\hat{m}}$ different configurations all belong to the same basin of attraction, since they only differ in the value of

Table 2. Column one: the probabilities $Y_{P,k}$, where $\Pi_P(Y_P)$ has support. Column two: the values of $Y_{P,k}$ for $P=2$, see also figure 4 and 5. Column three: the P-independent probabilities π_k that a random sample of the network has $Y_P = Y_{P,k}$. $\pi_k/(1-q)$ is a polynomial in $x=\frac{1}{2}q$. The table is infinite, so to truncate it somewhere we given only the first twenty entries, arranged according to decreasing values of $Y_{2,k}$. For $q<\frac{1}{2}$ they contain 99% of the probability in Π_P. The sum of coefficients of individual powers of q add up to one because Π_P in (5.10) is a normalised probability distribution.

$Y_{P,k}$	$Y_{2,k}$	$\pi_k/(1-q)$
1	1	$1+\frac{1}{2}q+\frac{1}{4}q^2$
$(\frac{3}{4})^P+(\frac{1}{4})^P$	$\frac{5}{8}$	$\frac{1}{6}q^3$
$2(\frac{1}{2})^P$	$\frac{1}{2}$	$\frac{1}{2}q+\frac{3}{8}q^2+\frac{1}{4}q^3+\frac{1}{8}q^4$
$(\frac{1}{2})^P+2(\frac{1}{4})^P$	$\frac{3}{8}$	$\frac{1}{4}q^2$
$2(\frac{3}{8})^P+2(\frac{1}{8})^P$	$\frac{5}{16}$	$\frac{1}{6}q^3+\frac{1}{4}q^4+\frac{1}{12}q^5$
$3(\frac{5}{16})^P+(\frac{1}{16})^P$	$\frac{19}{64}$	$\frac{1}{10}q^5$
$4(\frac{1}{4})^P$	$\frac{1}{4}$	$\frac{1}{8}q^2+\frac{13}{48}q^3+\frac{7}{32}q^4+\frac{1}{8}q^5$
$3(\frac{1}{4})^P+(\frac{1}{8})^P+2(\frac{1}{16})^P$	$\frac{27}{128}$	$\frac{1}{8}q^4$
$2(\frac{1}{4})^P+4(\frac{1}{8})^P$	$\frac{3}{16}$	$\frac{1}{8}q^3$
$5(\frac{3}{16})^P+(\frac{1}{16})^P$	$\frac{23}{128}$	$\frac{1}{12}q^6$
$4(\frac{3}{16})^P+4(\frac{1}{16})^P$	$\frac{5}{32}$	$\frac{1}{12}q^4+\frac{3}{16}q^5+\frac{1}{12}q^6+\frac{1}{24}q^7$
$6(\frac{5}{32})^P+2(\frac{1}{32})^P$	$\frac{19}{128}$	$\frac{1}{10}q^5+\frac{3}{20}q^6+\frac{1}{20}q^7$
$8(\frac{1}{8})^P$	$\frac{1}{8}$	$\frac{1}{48}q^3+\frac{17}{128}q^4+\frac{13}{96}q^5+\frac{1}{8}q^6$
$2(\frac{3}{16})^P+4(\frac{3}{32})^P+2(\frac{1}{16})+4(\frac{1}{32})^P$	$\frac{15}{128}$	$\frac{1}{24}q^5$
$6(\frac{1}{8})^P+4(\frac{1}{16})^P$	$\frac{7}{64}$	$\frac{1}{32}q^4+\frac{1}{16}q^5$
$9(\frac{7}{64})^P+(\frac{1}{64})^P$	$\frac{221}{2048}$	$\frac{1}{14}q^7$
$6(\frac{1}{8})^P+2(\frac{1}{16})^P+4(\frac{1}{32})^P$	$\frac{27}{256}$	$\frac{1}{16}q^5$
$4(\frac{1}{8})^P+8(\frac{1}{16})^P$	$\frac{3}{32}$	$\frac{1}{32}q^4$
$6(\frac{15}{128})^P+6(\frac{5}{128})^P+2(\frac{3}{128})^P+(\frac{1}{128})^P$	$\frac{1619}{16384}$	$\frac{1}{20}q^8$
$10(\frac{3}{32})^P+2(\frac{1}{32})^P$	$\frac{23}{256}$	$\frac{1}{24}q^6+\frac{1}{12}q^7$
$\dots$	$\dots$	$\dots$

their irrelevant variables. Any configuration, the relevant variables of which go through a cycle of period c, belongs to a basin of attraction containing $c2^{N-\hat{m}}$ different configurations, i.e. a basin having weight

$$W = c2^{-\hat{m}}. \tag{6.1}$$

From (6.1) we learn four things.

(i) The weight of a basin of attraction is proportional to the period c of the attractor. The constant of proportionality is given by the number of relevant variables $\hat{m}$ of the sample under consideration. This result is implicitly used in § 5, where also the probability for c was found for given $\hat{m}$.

(ii) The weight of a basin of attraction is a rational number and the denominator is a power of two.

(iii) For given $\hat{m}$, the smallest weight possible is

$$W_{\min}(\hat{m}) = 2^{-\hat{m}}. \tag{6.2}$$

It is realised by fixpoints in configuration space.

(iv) The largest weight possible for given $\hat{m}$ is a decreasing function of $\hat{m}$ satisfying

$$W_{\max}(\hat{m}) \leq 2\times 2^{-\hat{m}/2} \qquad \text{for } \hat{m}>7. \tag{6.3}$$

The proof of (6.3) falls into two parts. The first part goes as follows. From (6.1) it is clear that $2^{\hat{m}} W_{\max}(\hat{m}) = c_{\max}(\hat{m})$, where $c_{\max}(\hat{m})$ is the longest cycle possible with $\hat{m}$ relevant variables. $c_{\max}(\hat{m})$ is found by the following considerations.

Consider an information conserving loop of length L. Let it be equipped with an odd number of input functions 'not'. Then by a judicial choice of initial configuration for the variables on this loop their time development will describe a cycle of period $2L$. This is the longest period possible for variables on a loop of length L. If we have m_L loops of length L, $m_L > 1$, the longest period possible for all the Lm_L variables on these loops is still $2L$. If, however, we also have information conserving loops of other lengths, say L', then the longest period possible for $L+L'$ variables on two loops of lengths L and L' is the least common multiple of $2L$ and $2L'$: $\mathrm{LCM}(2L, 2L') = 2\,\mathrm{LCM}(L, L')$. This is easy to see, and easy to generalise to the result

$$c_{\max}(\hat{m}) = 2 \max\left\{ \mathrm{LCM}(L_1, L_2, \ldots, L_n) \,|\, L_i \in \mathbb{N}, \sum_i L_i = \hat{m} \right\}. \tag{6.4}$$

The second part of the proof of (6.3) consists in finding an upper bound for the RHS of (6.4). This is a problem of 'combinatorial optimisation' which is treated in the appendix.

In § 5 we calculated all the contributions to $g(W)$ and $\Pi_P(Y_P)$ from samples having $\hat{m} \leq 4$. We neglected all contributions from samples with $\hat{m} \geq 5$. The latter contribute to $g(W)$ only in the interval $[0, W_{\max}(5)]$ since $W_{\max}(\hat{m})$ is a decreasing function of $\hat{m}$. Consequently, samples with $\hat{m} \geq 5$ contribute to $\Pi_P(Y_P)$ only in the interval $[0, (W_{\max}(5))^{P-1}]$.

Thus we see that the expressions obtained for $g(W)$ and $\Pi_P(Y_P)$ in § 5 are exact for $W_{\max}(5) < W \leq 1$ and $W_{\max}(5)^{P-1} < Y_P \leq 1$. We have improved these results using a computer program to calculate $g(W)$ and $\Pi_P(Y_P)$ exactly in the intervals $0.0002 < W \leq 1$ and $(0.0002)^{P-1} < Y_P \leq 1$ corresponding to $\hat{m} \leq 23$.

Figure 1 shows $g(W)$ for $q = 0.2$, 0.5, 0.8 and 0.99. Its most characteristic feature is of course the delta function spectrum with support at $W = 1, \frac{3}{4}, \frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \ldots$, independent of q. We also see how $g(W)$ approaches $\delta(W)$ for $q \to 1$, though this is easier to see in figure 2, and even more so in figure 3. Figure 2 clearly shows the accumulation of points of support at $W = 0$. This accumulation makes it possible for $q = 1$ to be a critical point, as described in § 8, because it makes it possible to shift the total probability in $g(W)$ towards $W = 0$ as q increases, even though $g(0) = 0$ for all values of $q < 1$. Figure 3 shows the integrated probability

$$\mathrm{int}\, g(W) = \int_0^W \mathrm{d}W'\, g(W') \tag{6.5}$$

for $q = 0.2$, 0.5, 0.8 and 0.99. Here it is even clearer how the probability $g(W) \to \delta(W)$, hence $\mathrm{int}\, g(W) \to \theta(W)$, for $q \to 1$. The broken curve represents

$$\mathrm{int}\, g_\infty(W) = 1 - (1 - W)^{1/2} \tag{6.6}$$

corresponding to

$$g_\infty(W) = 1/[2(1 - W)^{1/2}] \tag{6.7}$$

which is the exact analytical result for Kauffman's model with $K = \infty$, and any finite value for p [15]. This is the only other case for which the model has been solved exactly. For these parameter values one is as far into its chaotic phase as one can possibly come, while for $K = 1$, q finite, one is in its frozen phase (see § 8 and

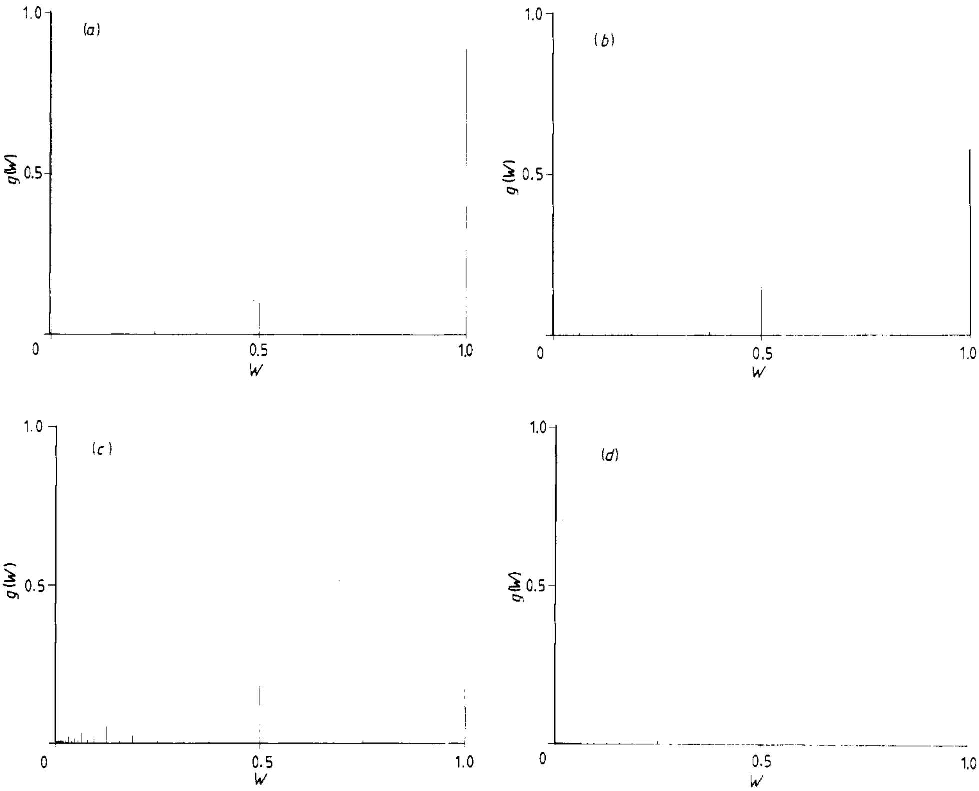

Figure 1. $g(W)$ for (*a*) $q=0.2$, (*b*) $q=0.5$, (*c*) $q=0.9$ and (*d*) $q=0.99$. g is a sum of delta functions. Its points of support do not depend on q, and accumulate at $W=0$. $g(W)\to\delta(W)$ for $q\to 1$.

[10, 12, 17, 23]). So it is not surprising that int $g_\infty(W)$ resembles no int $g(W)$, whatever the value of q.

Figures 4 and 5 show $\Pi(Y)$ for $q=0.2$, 0.5, 0.8 and 0.99. Like $g(W)$, $\Pi(Y)$ has a characteristic delta-function spectrum, the support of which is independent of q. The Π spectrum is denser than that of g because $Y=\Sigma_s W_s$ assumes more different values than does W_s. Figure 6 shows the integrated probability

$$\operatorname{int}\Pi(Y)=\int_0^Y \mathrm{d}Y'\,\Pi(Y') \tag{6.8}$$

for $q=0.2$, 0.5, 0.8 and 0.99. We see that $\operatorname{int}\Pi(Y)\to\theta(Y)$, hence $\Pi(Y)\to\delta(Y)$ for $q\to 1$. This is also seen in figures 4 and 5.

7. Exact and narrow bounds on expectation values

We have calculated narrow bounds on expectation values with respect to the exact probability distributions g and Π_P using our approximate expressions for these functions and our knowledge of how they approximate. Let $P(x)$ be a normalised probability distribution in the unit interval and let $p(x)$ be a non-negative function that

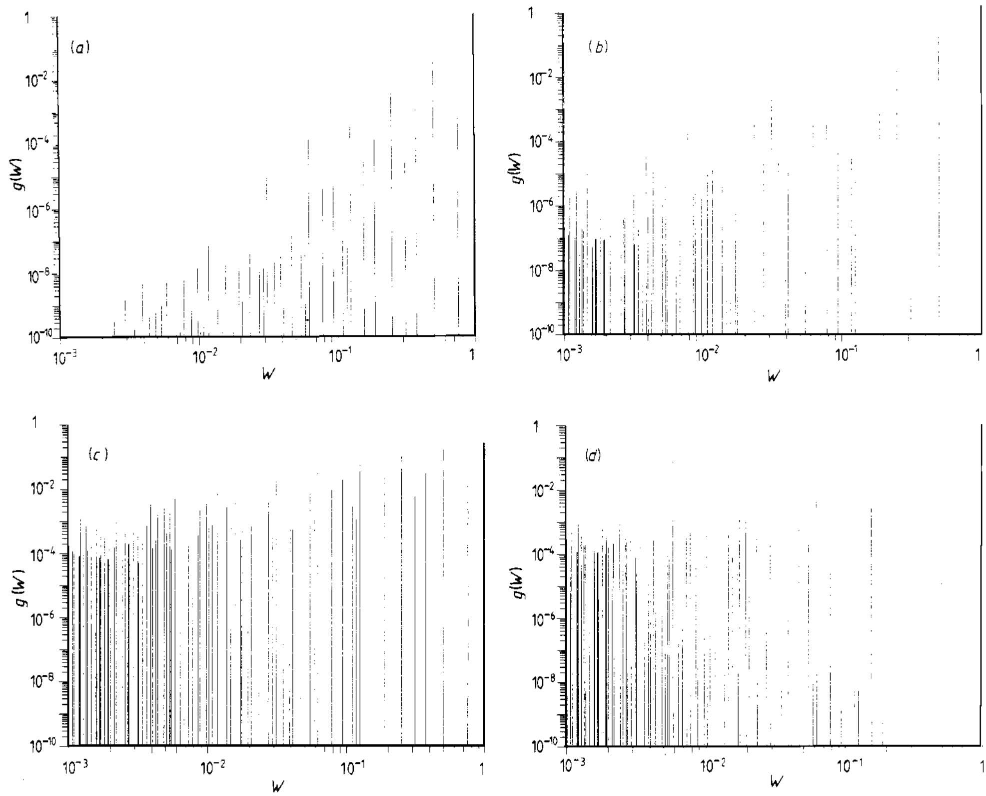

Figure 2. Same as figure 1, but with logarithmic axes.

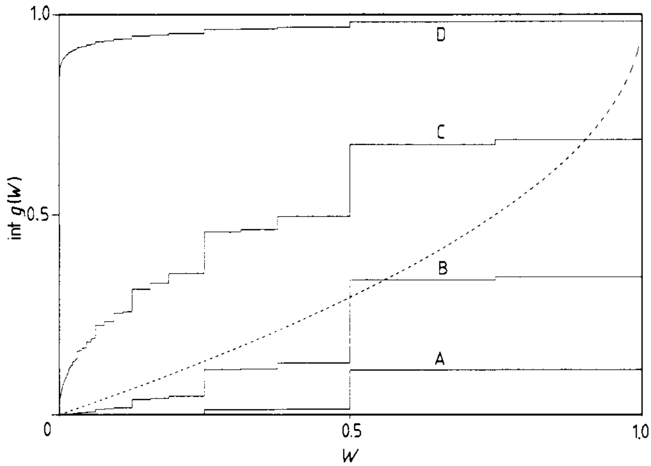

Figure 3. The integrated probability int $g(W) = \int_0^W \mathrm{d}W'\, g(W')$ for $q = 0.2$ (A), 0.5 (B), 0.8 (C) and 0.99 (D). int $g(W) \to \theta(W)$ for $q \to 1$. The broken curve is int $g(W) = 1-(1-W)^{1/2}$ for Kauffman's model with connectivity $K = \infty$ [14].

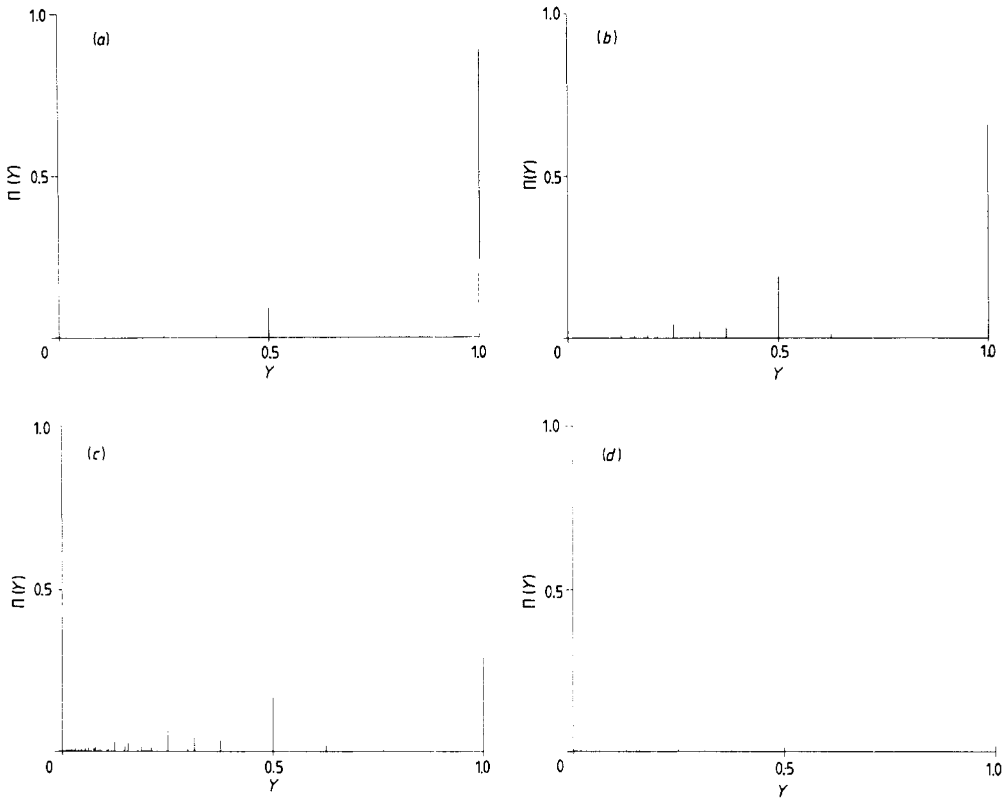

Figure 4. $\Pi(Y)$ for (*a*) $q=0.2$, (*b*) $q=0.5$, (*c*) $q=0.8$ and (*d*) $q=0.99$. Π is a sum of delta functions. Its points of support do not depend on q and accumulate at $Y=0$. $\Pi(Y)\to\delta(Y)$ for $q\to 1$.

approximates $P(x)$ from below:

$$\int_0^1 \mathrm{d}x\, P(x)=1 \tag{7.1}$$

$$0\leqslant p(x)\leqslant P(x) \qquad \forall x\in[0,1] \tag{7.2}$$

$$p(x)=P(x) \qquad \forall x>x_0\in[0,1]. \tag{7.3}$$

Define for later use

$$\varepsilon=1-\int_0^1 \mathrm{d}x\, p(x)=\int_0^{x_0}\mathrm{d}x(P(x)-p(x)). \tag{7.4}$$

Then it is easy to show for any non-negative function $f(x)$ that

$$\begin{aligned}\int_0^1 \mathrm{d}x\, p(x)f(x)&\leqslant\int_0^1 \mathrm{d}x\, P(x)f(x)\\ &\leqslant\int_0^1 \mathrm{d}x\, p(x)f(x)+\sum \max_{0\leqslant x\leqslant x_0} f(x).\end{aligned} \tag{7.5}$$

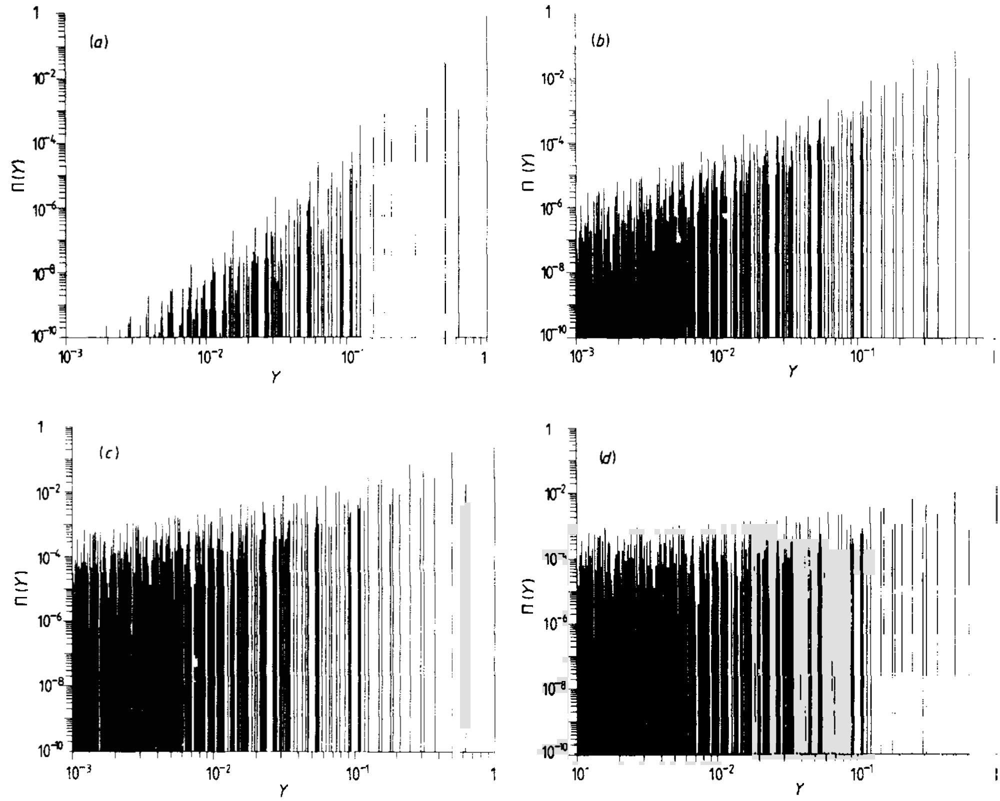

Figure 5. Same as figure 4, but with logarithmic axes.

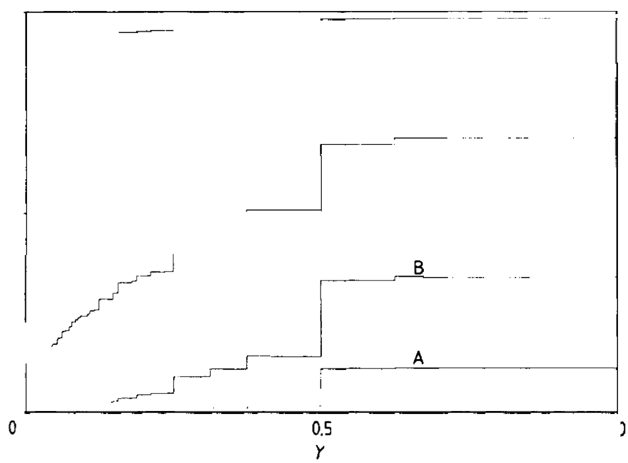

Figure 6. The integrated probability $\mathrm{int}\,\Pi(Y)=\int_0^Y \mathrm{d}Y'\,\Pi(Y')$ for $q=0.2$ (A), 0.5 (B), 0.8 (C) and 0.99 (D). $\mathrm{int}\,\Pi(Y)\to\theta(Y)$ for $q\to 1$.

Using (7.5) one finds that

$$\bar{Y} = \bar{Y}_2 = \int_0^1 \mathrm{d}W g(W) W = \int_0^1 \mathrm{d}Y \Pi_2(Y) Y \tag{7.6}$$

satisfies

$$\begin{aligned} \bar{Y}^{(23)} < \bar{Y} < \bar{Y}^{(23)} + q^{24} W_{\max}(23) \\ \leqslant \bar{Y}^{23} + 2\times 10^{-4} \qquad \text{for } q \leqslant 1. \end{aligned} \tag{7.7}$$

A similar result is easily obtained for $\bar{Y}_P$, P arbitrary. For

$$\overline{Y^2} = \int_0^1 \mathrm{d}Y \Pi(Y) Y^2 \tag{7.8}$$

we have found

$$\begin{aligned} \overline{Y^{2(23)}} < \overline{Y^2} < \overline{Y^{2(23)}} + q^{24} W_{\max}(23) \\ \leqslant \overline{Y^{2(23)}} + 4\times 10^{-8} \qquad \text{for } q \leqslant 1 \end{aligned} \tag{7.9}$$

and conclude that $\sigma^2(Y) = \overline{Y^2} - \bar{Y}^2$ as upper bound has

$$\sigma^2(Y)^{(23)} + 4\times 10^{-8}$$

and as lower bound has

$$\sigma^2(Y)^{(23)} - 4\times 10^{-4}.$$

The bounds on expectation values derived in this section are more narrow than the tip of the pen used to draw figure 7. This figure then shows exact results. The upper curve in the figure is $\overline{Y(q)}$, the lower curve is $\overline{Y_4(q)}$. Between these two curves are two other curves practically, but not exactly, on top of each other, $\overline{Y_3(q)}$ and $\overline{Y^2(q)}$. The vanishing of all these expectation values at $q = 1$ is the subject of the next section.

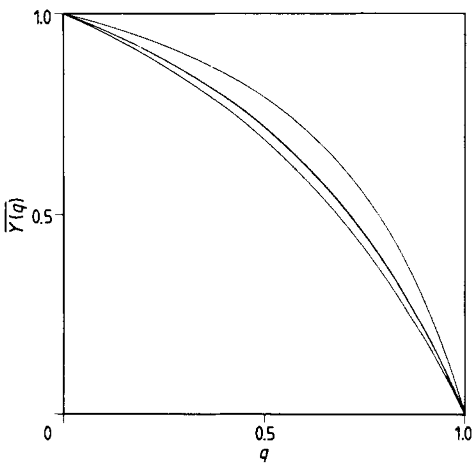

Figure 7. $\overline{Y(q)}$, $\overline{Y_3(q)}$, $\overline{Y_4(q)}$ and $\overline{Y^2(q)}$ against q. $\overline{Y(q)}$ is the upper curve, $\overline{Y_4(q)}$ the lower curve. In between are the two other curves practically, but not exactly, on top of each other.

8. The critical point at $q = 1$

In § 4 we obtained $Q(m)$ in the thermodynamic limit $N \to \infty$. In this section we consider a second limit, $q \to 1$. We find that

$$g(W) = \delta(W) \qquad \text{for } q = 1 \tag{8.1}$$

$$\Pi_P(Y_P) = \delta(Y_P) \qquad \text{for } q = 1. \tag{8.2}$$

This follows from the normalisation condition

$$\int_0^1 \mathrm{d}W g(W) = 1 \tag{8.3}$$

$$\lim_{q \to 1} \int_\varepsilon^1 \mathrm{d}W g(W) = 0 \qquad \forall \varepsilon > 0 \tag{8.4}$$

and identical equations for Π_P. Equation (8.4) is a consequence of § 6, point (iv), and the appendix: given ε one chooses $\hat{m}(\varepsilon)$ such that $W_{\max}(\hat{m}(\varepsilon)) < \varepsilon$. Then any distribution $(m_L)_{L=1,2,\ldots}$ of loops with $\hat{m} > \hat{m}(\varepsilon)$ contributes only to $g(W)$ for $0 < W < \varepsilon$. Consequently

$$\begin{aligned}\int_\varepsilon^1 \mathrm{d}W g(W) &\leq \sum_{\substack{m \\ \hat{m} < \hat{m}(\varepsilon)}} Q(m) \\ &= (1-q) \sum_{\substack{m \\ \hat{m} < \hat{m}(\varepsilon)}} q^{\hat{m}} \left(\prod_{L=1}^{\infty} m_L! L^{m_L} \right)^{-1}\end{aligned} \tag{8.5}$$

where the last identity follows from (5.2). Since the coefficient to $1 - q$ in (8.5) remains finite for $q \to 1$, (8.4) follows.

Equation (8.5) indicates that any expectation value with respect to $g(W)\,\mathrm{d}W$ approaches its limiting value for $q \to 1$ in a manner proportional to $1 - q$. This is indeed what we find for the moments $\bar{Y}$, $\overline{Y_3}$ and $\overline{Y_4}$, and for the second moment $\overline{Y^2}$ of $\Pi(Y)$ (see figure 7).

It is not difficult to understand why for $q \to 1$ only basins with vanishing weights W occur: equation (3.11) shows that in the thermodynamic limit the connection graph contains loops of all lengths, finite loops having vanishing probability. The very long loops, which can be information conserving for $q \to 1$, split configuration space into very many tiny basins of attraction. The average length $\bar{L}$ of information conserving loops expresses this picture in a quantitative way:

$$\bar{L} = \sum_L \bar{m}_L L \left(\sum_L \bar{m}_L \right)^{-1} = \frac{-q}{(1-q)\ln(1-q)}. \tag{8.6}$$

In (8.6) we have used $\bar{m}_L = \Sigma_m\, Q(m) m_L = q^2/L$.

$\bar{L}$ diverges for $q \to 1$ as does the susceptibility of a magnetic system, when the temperature approaches the critical temperature.

Kauffman's model also has an order parameter analogous to a magnetisation. It is the relative size of the 'stable core', the latter being the subset of variables in a sample that acquire a value for time $\to \infty$ which does not depend upon the initial configuration. Let us calculate its size $s(\infty)$: the variables which after one time step acquire a value that they maintain thereafter, irrespective of the initial configuration, are those that are updated with the constant functions 'zero' and 'one'. Their relative number $s(1)$ equals $1-q$:

$$s(1)=1-q. \tag{8.7}$$

After two time steps the relative number $s(2)$ of variables having a value they will maintain thereafter, irrespective of the initial configurations, is

$$s(2)=s(1)+(1-s(1))s(1) \tag{8.8}$$

where in the last term $1-s(1)$ is the relative number of variables not stable after one time step, and the factor $s(1)$ is the probability that such a variable receives its input from a variable stable after one time step, and thus becomes stable itself after two time steps. Let $s(t)$ denote the relative number of variables which after t time steps have acquired a value which they keep thereafter, and which does not depend on the initial configuration. Then (8.7) and (8.8) generalise to

$$s(t+1)=s(t)+(1-s(t))\frac{s(t)-s(t-1)}{1-s(t-1)} \tag{8.9}$$

where $[s(t)-s(t-1)]/[1-s(t-1)]$ is the probability that a variable becomes stable in the $(t+1)$th time step by receiving its input from a variable that became stable in the tth time step. The unique solution to (8.9) is

$$s(t)=1-q^t. \tag{8.10}$$

Thus we see that the size of the stable core is

$$\lim_{t\to\infty} s(t)=\begin{cases}1 & \text{for } q<1\\ 0 & \text{for } q=1.\end{cases} \tag{8.11}$$

In analogy with magnetic systems we may say that Kauffman's model with $K=1$ has a critical point at $q=1$, with a diverging 'susceptibility' $\bar{L}$ and a discontinuous 'magnetisation' $s(\infty)$, or better $1-s(\infty)$. This behaviour much resembles that of one-dimensional magnetic systems. They share with Kauffman's model with $K=1$ the properties that they are exactly solvable, and the critical point is located at one extreme of the parameter interval, be it temperature, critical at zero, or q, critical at one.

9. Finite N

So far we have concentrated on the thermodynamic limit $N\to\infty$. In this section we consider finite networks. We do so both from a general interest in finite-N behaviour, and to obtain results directly comparable to results from numerical simulations of necessarily finite networks.

Equation (4.1) gives $Q(m)$ for any finite N. It differs little from its form in the thermodynamic limit, and the derivations of $g(W)$ and $\Pi_P(Y_P)$ in § 6 and their momenta in § 7 are unchanged in the case of finite N, except $\hat{m}\leq N$ must be satisfied. Figure 8(a) shows the N dependence of $\bar{Y}$ for various values of q. The case $q=\frac{1}{2}$ was simulated numerically in [14] with results agreeing with the exact results presented here. The case of $q=1$, critical for $N=\infty$, may be compared with the supposedly critical case $K=2$, $q=\frac{1}{2}$ in figure 1(a) in [14]. There it could not be decided whether or not $\bar{Y}\to 0$ for $N\to\infty$ because $N=240$ was the maximum size system that could be simulated, and $\sigma^2(Y)$ remained finite, though it too should go to zero if $\bar{Y}$ does.

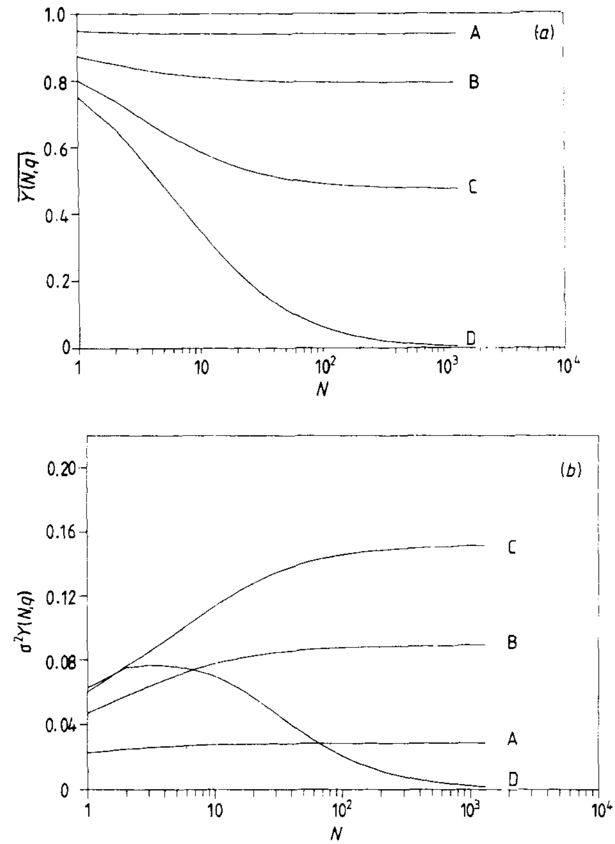

Figure 8. (a) $\bar{Y}$ as a function of finite network size N for $q=0.2$ (A), 0.5 (B), 0.8 (C) and the case $q=1$ (D), which is critical for $N=\infty$. (b) The variance $\sigma^2(Y)=\overline{Y^2}-\bar{Y}^2$ as a function of N for $q=0.2$ (A), 0.5 (B), 0.8 (C) and the critical case $q=1$ (D).

Figure 8(b) in the present paper shows that the problem in [14] could be that N was not sufficiently large. Figure 8(b) shows that, in the critical case $K=1$, $q=1$, $\sigma^2(Y)$ first increases with N and then decreases to zero.

Finally, figure 9(a)–(c) show $\overline{Y^2}$, $\overline{Y_3}$ and $\overline{Y_4}$ plotted against $\bar{Y}$ for various values of N and q. These figures should be compared with similar figures in [14] where the moments plotted against each other were evaluated by numerical simulation of the networks, and the parameters varied were N and K instead of N and q varied here.

The figures are rather similar: in both cases the relationship between moments are close but not equal to the theoretical relationship obeyed by the Sherrington–Kirkpatrick model for spin glasses [24–26]. We find a somewhat wider range of differences from the spin-glass result than was found in [14].

10. Conclusion

We have seen that the variables of Kauffman's model with connectivity $K=1$ fall into three classes: the first class is made up of the variables we have called 'relevant', the variables placed on 'information conserving' loops in the connection graph. The initial

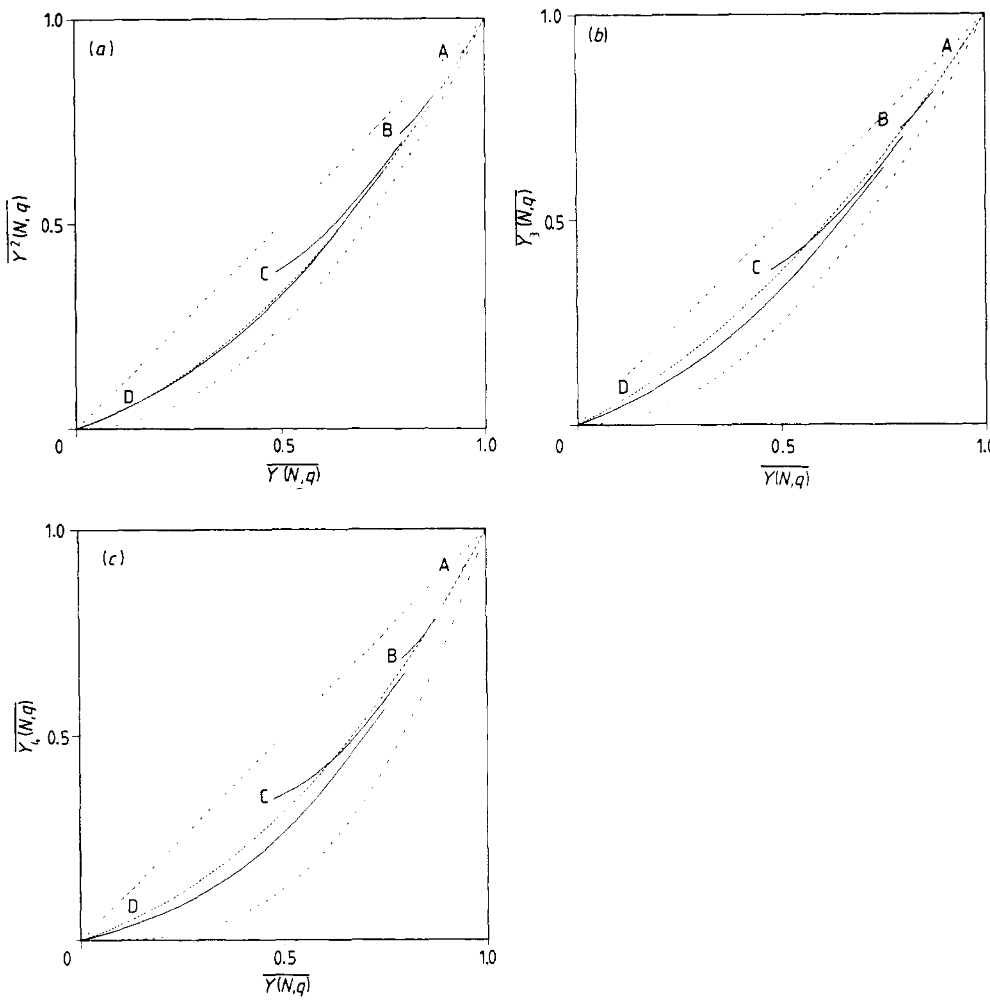

Figure 9. (*a*) Full curves: $\overline{Y^2}$ against $\bar{Y}$ for N varying from 1 to ∞ (making $\bar{Y}$ decrease) and $q=0.2$ (A), 0.5 (B), 0.8 (C) and 1 (D). Outer broken curves represent bounds $\bar{Y}^2 < \overline{Y^2} < \bar{Y}$ that are always satisfied. The central broken curve represents the equation $\overline{Y^2} = \frac{1}{3}(\bar{Y} + 2\bar{Y}^2)$ satisfied by the Sherrington–Kirkpatrick model [24–26]. (*b*) $\bar{Y}_3$ against $\bar{Y}$. Outer broken curves are bounds $\bar{Y}^2 < \bar{Y}_3 < \bar{Y}$, central broken curve the equation $\bar{Y}_3 = \frac{1}{2}\bar{Y}(1+\bar{Y})$ satisfied by the SK model. (*c*) $\bar{Y}_4$ against $\bar{Y}$. The bounds are $\bar{Y}^3 < \bar{Y}_4 < \bar{Y}$, and the result for the SK model $\bar{Y}_4 = \frac{1}{6}\bar{Y}(1+\bar{Y})(2+\bar{Y})$.

values of the relevant variables are necessary and sufficient to determine the limit cycle of any configuration. The second class we have paid little attention to, because it consists of 'dependent' variables, variables which after a finite time describe limit cycles determined by the limit cycles of the relevant variables, and differing at most by a phase shift from the latter. These 'dependent' variables are placed on 'information conserving' trees rooted in 'information conserving' loops in the connection graph. The third class is the complement to the first two classes. It consists of the variables which after a finite time acquire a constant value which is the same for any initial configuration. The third class is called 'the stable core'. Variables in it either receive input via constant functions, or have 'ancestors' in the connection graph that do so.

We found the stable core has measure one, except at a critical point, where it has measure zero.

We have seen that a sample of the model with $\hat{m}$ relevant variables contributes to the probability density $g(W)$ only in the interval $W_{\min}(\hat{m}) < W < W_{\max}(\hat{m})$ and to $\Pi_P(Y_P)$ only in the interval $W_{\min}(m)^{P-1} < Y_P < W_{\max}(m)^{P-1}$. Since $W_{\max}(\hat{m}) \to 0$ for $\hat{m} \to \infty$, in any interval not including zero only a finite number of samples contribute to the values of g and Π_P. They can therefore be computed exactly in such an interval, which we did for $0.0002 < W < 1$ and $0.0002 < Y_2 < 1$. For the same reason, g and Π_P are sums of delta functions.

We have used these results to describe a critical point, which is found to somewhat resemble the critical point at zero temperature possessed by one-dimensional spin systems. We have also described finite-size effects and maybe increased our understanding of finite-size effects at the critical point for connectivities $K > 1$ as well. Finally, we mention that Kauffman's model with any connectivity has the property that the stable core is of measure one at the critical point and in the entire frozen phase [27]. The complement to the stable core forms an effective network with effective connectivity $K' = 1$. For this reason we believe that the exact solution presented here can be made the starting point for approximation schemes that may describe Kauffman's model with any connectivity in the frozen phase and even slightly into the chaotic phase, with little modification.

Acknowledgments

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Appendix

Define

$$S(m) = \max\left\{ \text{LCM}(L_1, L_2, \ldots, L_n) \,\middle|\, \sum_{i=1}^{n} L_i = m,\ L_i \in \mathbb{N} \right\}$$

where LCM(...) means 'least common multiple of ...'. In this appendix we prove that

$$S(m) \leqslant 2^{m/2} \qquad \text{for } m > 7. \tag{A1}$$

For large values of m this inequality is far from saturated, but it suffices for our purposes.

Equation (A1) follows from the fact that

$$S(m) \text{ is an even number for } m > 15 \tag{A2}$$

and, as a consequence of this and direct inspection for $7 < m < 14$, that

$$S(m+2) \leqslant 2S(m) \qquad \text{for } m > 6. \tag{A3}$$

Let us prove this last statement first: we observe that $S(m)$ is a non-decreasing function of m: let $m = L_1 + L_2 + \ldots + L_m$ be a partitioning realising $S(m)$. Supplemented with 1 it is also a partitioning of $m+1$. So $S(m+1) \geqslant L_1 L_2 \ldots L_n 1 = S(m)$. Now let

$m = 2^k + L_2 + L_3 + \ldots + L_n$ be a partitioning of m realising $S(m)$. We have assumed (A2) to hold, so $k \geq 1$ for $m > 15$. Define

$$m' = \begin{cases} L_2 + L_3 + \ldots + L_n = m - 2 & \text{for } k = 1 \\ 2^{k-1} + L_2 + \ldots + L_n = m - 2^{k-1} & \text{for } k > 1. \end{cases} \tag{A4}$$

Then

$$S(m') \leq S(m-2). \tag{A5}$$

From the partitionings of m' in (A4) it follows that

$$S(m') > \tfrac{1}{2} S(m)$$

which, with (A5), gives (A3).

We have not found a short proof of (A2), so we only outline the stages of our long proof.

One can show that any natural number m has a partitioning into mutually prime terms realising $S(m)$, i.e. the terms of the partitioning have $S(m)$ as least common multiple. This lemma is easily sharpened: all terms of the partitioning may be chosen to be powers of primes, the primes being different for different terms or equal to one:

$$\begin{aligned} &S(m) = p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n} \\ &m \leq p_1^{k_1} + p_2^{k_2} + \ldots + p_n^{k_n} \\ &1 < p_1 < p_2 < \ldots < p_n \qquad p_i \text{ prime number.} \end{aligned} \tag{A6}$$

One proceeds to prove that either $p_1 = 2$ or $k_1 = k_2 = \ldots = k_n = 1$.

Proof. Assume $p_1 > 2$ and $k_i > 2$. Choose k_0 such that $2^{k_0 - 1} < p_i < 2^{k_0}$. Then

$$p_i < 2^{k_0} < 2p_i. \tag{A7}$$

Since $2 \leq p_i - 1$ and $p_i \leq p_k^{k_i - 1}$ we have $2p_i < p_i^{k_i} - p_i^{k_i - 1}$, which inserted in (A7) gives

$$p_i < 2^{k_0} < p_i^{k_i} - p_i^{k_i - 1}. \tag{A8}$$

From (A8) it follows that

$$\begin{aligned} &2^{k_0} p_i^{k_i - 1} > p_i^{k_i} \\ &2^{k_0} + p_i^{k_i - 1} < p_i^{k_i} \end{aligned} \tag{A9}$$

i.e. there is a partitioning of m which, instead of the term $p_i^{k_i}$, has two terms 2^{k_0} and $p_i^{k_i - 1}$, and which has a least common multiple of its terms that exceeds $S(m)$, in contradiction with the definition of $S(m)$.

Subsequently one assumes that

$$\begin{aligned} &S(m) = p_1 p_2 \ldots p_n \\ &m \leq p_1 + p_2 + \ldots + p_n \\ &2 < p_1 < p_2 < \ldots < p_n \qquad p_i \text{ prime} \end{aligned} \tag{A10}$$

and finds that the sequence $p_1, p_2, \ldots, p_n$ contains all primes between 3 and p_n.

Proof. Assume p_{i-1} and p_i are not consecutive primes. Choose a prime p such that $p \leq p_i - 2$ and $p > \max(p_{i-1}, \frac{1}{2}p_i)$. Then $p + 2 \leq p_i$ while $2p > p_i$, i.e. if in (A10) p_i is replaced by the two terms 2 and p a product larger than $S(m)$ is obtained in contradiction with its definition. Now repeat this proof with $p_{i-1} = 2$, $p_i = p_1$ and find $p_1 = 3$.

Finally we prove $p_n < 11$: assume $p_n \geqslant 11$. Then 3 and 11 contribute to $S(m)$ with a factor 33. Another partitioning of m is obtained by replacing $3+\ldots+11\ldots$ by $1+2^2+3^2+\ldots$ which contributes a factor $36>33$ to $S(m)$; again a contradiction.

In summary: $S(m)$ is even except for $S(3)=3$, $S(8)=15$ and $S(15)=105$.

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